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H. E. RORSCHACH, JR.
MELVIN A. HERLIN

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Abstract

A method for measuring the electrical resistivity of a cylindrical specimen of metal is described. The specimen is mounted at the center of a pair of concentric coils whose complex mutual inductance is measured. The field equations for this geometry are solved so that mutual inductance values may be used to yield resistivity values. The method is particularly useful for low temperature measurements because no electrical or other connection needs to be made to the specimen, and because the sample size and shape are suitable both for ease of preparation and for other measurements.

AN INDUCTION METHOD OF MEASURING ELECTRICAL RESISTIVITY

I. Introduction

An induction method of measuring the electrical conductivity of bulk metallic specimens has recently been developed in connection with some measurements on the low temperature resistivity of magnesium (1). The method has two distinct advantages: (a) No connections to the specimen are necessary; thus the contact potentials and the thermal emf's are eliminated and the cryogenic apparatus necessary for low temperature measurements is simplified; (b) bulk specimens of material may be used rather than drawn wires. The use of bulk specimens simplifies the metallurgical processing of the samples to be studied, including control of the crystalline state and the introduction of impurities. In addition, resistivity measurements may be made on samples of size and shape appropriate for specific heat and thermal conductivity measurements.

A mutual inductance coil, consisting of a primary coil and a secondary coil wound on a cylindrical form, contains a cylindrical core of the material whose resistivity is to be determined. The real and the imaginary components of the mutual inductance of the primary-secondary system are measured on an ac mutual inductance bridge. The arrangement is shown schematically in Fig. 1. The resistivity of the core is computed from the mutual inductance values.

Although a mutual inductance method of this type has been used previously in the study of the penetration depth in superconductors, the method was not sufficiently refined to measure conductivities. The purpose of this report is to carry out the calculation of the resistivity of the sample core from the measured mutual inductance values and the geometry of the system.

II. Electromagnetic Equations

The starting point is Maxwell's equations (in rationalized mks units)

$$\nabla \times \vec{E} = -\dot{\vec{B}}$$

$$\nabla \times \vec{H} = \vec{J}$$

$$\nabla \cdot \vec{B} = 0$$

and the constitutive relations

$$\vec{B} = \mu_0 \vec{H} \quad \vec{J} = \sigma \vec{E}$$

These equations are given in quasi-static form, since the frequency is so low (33 1/3 cps) that the displacement current may be neglected.

The mutual inductance of a system is defined as the secondary flux linkage per unit of primary current

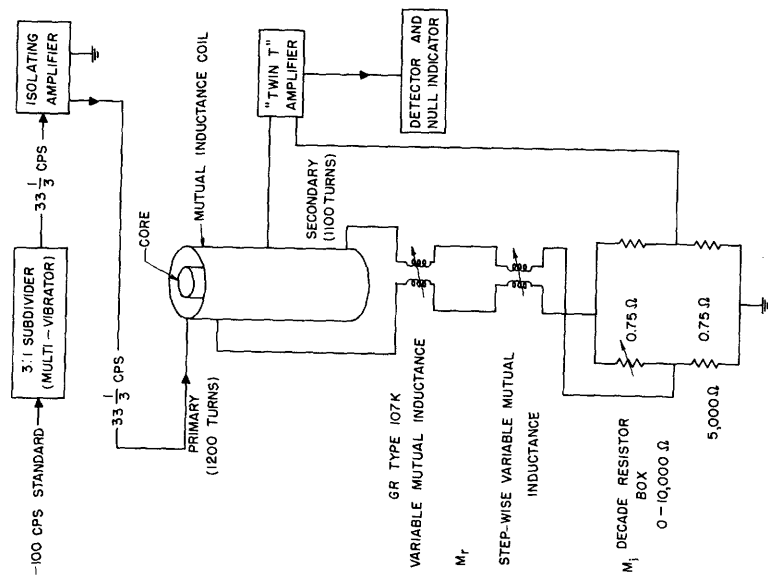


Fig. 1

Block diagram of the mutual inductance system used for resistivity measurements.

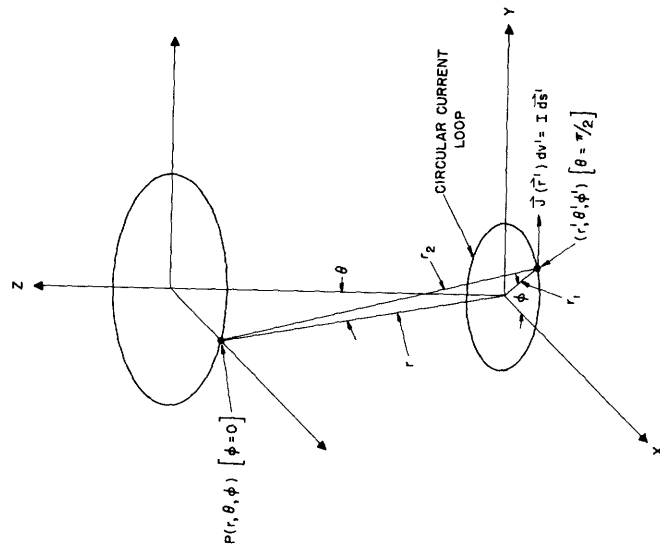


Fig. 2

Construction for the expansion of the vector potential in spherical harmonics.

$$M = \frac{N\phi}{I}$$

For the purpose of calculating mutual inductance, expressions for the magnetic flux will be required

$$\phi = \int_A \vec{B} \cdot \vec{n} dA$$

or, if we introduce the vector potential, $\vec{B} = \nabla \times \vec{A}$, we have

$$\begin{aligned} \phi &= \int_A (\nabla \times \vec{A}) \cdot \vec{n} dA \\ &= \oint \vec{A} \cdot d\vec{s} \end{aligned}$$

by Stoke's theorem. Thus, the flux through an area may be determined by calculating the line integral of the vector potential around the boundary of the area. We shall use the vector potential throughout this paper.

If we now choose $\nabla \cdot \vec{A} = 0$, we may write the equation for \vec{A}

$$\nabla^2 \vec{A} = -\mu_0 \vec{J} \quad (1)$$

with

$$\nabla^2 \vec{A} = -\nabla \times \nabla \times \vec{A}$$

To illustrate the use of the vector potential, we shall derive the vector potential for a uniform magnetic field in the z-direction. Since $\vec{B} = \nabla \times \vec{A}$

$$(\nabla \times \vec{A})_z = B_0$$

and all other components are zero. Thus

$$\frac{1}{r} \frac{\partial}{\partial r} (r A_\theta) - \frac{1}{r} \left(\frac{\partial A_r}{\partial \theta} \right) = B_0$$

The simplest choice of \vec{A} is to choose $A_z = A_r = 0$, and $A_\theta = f(r)$. Then the equation for A_θ can be easily solved to give

$$A_\theta = B_0 \frac{r}{2} + \frac{\text{constant}}{r}$$

If the field includes $r = 0$, the constant must equal zero to maintain a finite potential (2).

For convenience in calculation, the expansion of the vector potential as a series of magnetic multipoles will be required. The vector potential due to a given current distribution can be calculated (3) from the following relation (for sufficiently low frequencies):

$$\vec{A}(P) = \frac{\mu_0}{4\pi} \int_{V'} \frac{\vec{J}(\vec{r}')}{r_2} dv'$$

with $\vec{A}(P)$ the vector potential at point P, $\vec{J}(\vec{r}')$ the current density at the point \vec{r}' , and r_2 the distance from \vec{r}' to P (see Fig. 2). This equation is the solution of Eq. 1 for \vec{A} , where Green's function for the three-dimensional case (4) is used. The quantity $1/r_2$ is then expanded in a series of spherical harmonics (5), yielding

$$\frac{1}{r_2} = \sum_{m, n} (2 - \delta_{0m}) \frac{(n - m)!}{(n + m)!} P_n^m(\cos\theta) P_n^m(\cos\theta') \cos m(\phi - \phi') \frac{r'^n}{r^{n+1}}$$

This expansion gives directly the familiar multipole expansion for the magnetic field. If P is chosen as shown in Fig. 2

$$\begin{aligned} \frac{1}{r_2} &= \sum_{m, n} (2 - \delta_{0m}) \frac{(n - m)!}{(n + m)!} P_n^m(\cos\theta) P_n^m(0) \cos m\phi' \frac{r'^n}{r^{n+1}} \\ &= \frac{1}{r} + \frac{r'}{r^2} \sin\theta \cos\phi' + \frac{1}{2} \frac{r'^2}{r^3} [1 - 3 \sin^2\theta \sin^2\phi'] \\ &\quad + \frac{1}{2} \frac{r'^3}{r^4} [5 \sin^3\theta \cos^3\phi' - 3 \sin\theta \cos\phi'] + \dots \end{aligned}$$

and thus

$$\vec{A}(P) = \frac{\mu_0}{4\pi} \int_{V'} \vec{J}(\vec{r}') \left[\frac{1}{r_2} \right] dv' = \vec{A}^{(0)} + \vec{A}^{(1)} + \vec{A}^{(2)} + \vec{A}^{(3)} + \dots$$

with

$$\text{monopole: } \vec{A}^{(0)} = \frac{\mu_0}{4\pi} \int_{V'} \vec{J}(\vec{r}') \frac{1}{r} dv'$$

$$\text{dipole: } \vec{A}^{(1)} = \frac{\mu_0}{4\pi} \int_{V'} \vec{J}(\vec{r}') \frac{r'}{r^2} \sin\theta \cos\phi' dv'$$

$$\text{quadrupole: } \vec{A}^{(2)} = \frac{\mu_0}{4\pi} \int_{V'} \vec{J}(\vec{r}') \cdot \frac{1}{2} \frac{r'^2}{r^3} [1 - 3 \sin^2\theta \sin^2\phi'] dv'$$

$$\text{octapole: } \vec{A}^{(3)} = \frac{\mu_0}{4\pi} \int_{V'} \vec{J}(\vec{r}') \frac{1}{2} \frac{r'^3}{r^4} [5 \sin^3\theta \cos^3\phi' - 3 \sin\theta \cos\phi'] dv'$$

The expansion in spherical harmonics thus yields the vector potential for a magnetic field as a series of terms representing the vector potential for the various multipole components of the field. For closed current filaments carrying a current I

$$\vec{J}(\vec{r}') dv' \rightarrow I(\vec{r}') d\vec{s}' = I d\vec{s}'$$

so that

$$\vec{A}^{(0)} = \frac{\mu_0 I}{4\pi} \int_{s'} \frac{d\vec{s}'}{r}$$

$$\vec{A}^{(1)} = \frac{\mu_0 I}{4\pi} \frac{\sin \theta}{r^2} \int_{s'} r' \cos \phi' d\vec{s}'$$

and so on.

In a later section of this report, we shall be interested in the vector potential due to a circular current loop, as shown in Fig. 2. We must therefore integrate the expression for the vector potential over the current distribution.

For $\vec{A}^{(0)}$ we may easily perform the integration. Since

$$\int_{s'} d\vec{s}' = 0$$

we have $\vec{A}^{(0)} = 0$.

For the calculation of $\vec{A}^{(1)}$, we put

$$d\vec{s}' = |r' d\phi'| \left[-\vec{i}_x \sin \phi' + \vec{i}_y \cos \phi' \right]$$

with \vec{i}_x and \vec{i}_y the unit vectors along the x and y axes. Then

$$\begin{aligned} \vec{A}^{(1)} &= \frac{\mu_0 I r'}{4\pi r^2} \sin \theta \int_{s'} \cos \phi' d\vec{s}' \\ &= \frac{\mu_0 I r'}{4\pi r^2} \sin \theta \int_0^{2\pi} \cos \phi' r' \left[-\vec{i}_x \sin \phi' + \vec{i}_y \cos \phi' \right] d\phi' \end{aligned}$$

Thus

$$\vec{A}^{(1)} = \frac{\mu_0 I r'^2}{4\pi r^2} \sin \theta \vec{i}_y$$

and if $I \pi r'^2 = m$, then

$$\vec{A}^{(1)} = \frac{\mu_0 m \sin \theta}{4\pi r^2} \vec{i}_y \quad (2)$$

Since the current distribution is symmetric, we may take P as an arbitrary point on the circle shown in Fig. 2, in which case $\vec{i}_y \rightarrow \vec{i}_\phi$.

To calculate $\vec{A}^{(2)}$ we recall

$$\vec{A}^{(2)} = \frac{\mu_0}{4\pi} \int_{S'} \frac{1}{2} \frac{I r'^2}{r^3} [1 - 3 \sin^2 \theta \sin^2 \phi] d\vec{s}'$$

and using the expression for $d\vec{s}'$ developed above, we obtain

$$\vec{A}^{(2)} = \frac{\mu_0}{8\pi} I \frac{r'^2}{r^3} \left[\int_{S'} d\vec{s}' - 3 \sin^2 \theta \int_{S'} \sin^2 \phi' d\vec{s}' \right] = 0$$

and $\vec{A}^{(2)}$ is therefore zero.

Similarly, we may calculate $\vec{A}^{(3)}$. Since

$$\vec{A}^{(3)} = \frac{\mu_0}{4\pi} \int_{S'} \frac{1}{2} I \frac{r'^3}{r^4} [5 \sin^3 \theta \cos^3 \phi' - 3 \sin \theta \cos \phi'] d\vec{s}'$$

then

$$\vec{A}^{(3)} = \frac{\mu_0 I r'^3}{8\pi r^4} \left[5 \sin^3 \theta \int_{S'} \cos^3 \phi' d\vec{s}' - 3 \sin \theta \int_{S'} \cos \phi' d\vec{s}' \right]$$

or

$$\vec{A}^{(3)} = \frac{3}{8\pi} \frac{\mu_0 I \pi r'^4}{r^4} \sin \theta \left[\frac{5}{4} \sin^2 \theta - 1 \right] \vec{i}_y$$

III. The Mutual Inductance Calculation

A. The Case of an Infinite Core in an Infinite Coil

As the preliminary step to the solution of the problem, we solve the case of an infinite cylindrical core in an infinite cylindrical coil. We take the axis of the coil-core system as the z-axis and set the radius of the core equal to a. We must solve

$$\nabla^2 \vec{A} = -\mu_0 \vec{J} \quad (3)$$

To eliminate \vec{J} , we use $\nabla \times \vec{E} = -\dot{\vec{B}}$. Assuming a complex exponential time dependence for the magnetic field, $\vec{B} = \vec{B}_0 \exp[j\omega t]$, we obtain

$$\nabla \times \vec{E} = -j\omega \vec{B}_0 \exp[j\omega t]$$

or, if

$$\vec{B} = \nabla \times \vec{A}$$

then

$$\nabla \times \vec{E} = -j\omega \nabla \times \vec{A}$$

Thus we may select $\vec{E} = -j\omega \vec{A}$; and using $\vec{J} = \sigma \vec{E}$, we obtain $\vec{J} = -j\omega \sigma \vec{A}$. Thus Eq. 3 reduces to

$$\nabla^2 \vec{A} = -\nabla \times \nabla \times \vec{A} = j\mu_o \omega \sigma \vec{A}$$

or

$$\nabla \times \nabla \times \vec{A} + j\mu_o \omega \sigma \vec{A} = 0$$

In cylindrical coordinates, this equation becomes

$$\frac{d^2 A_\theta}{dr^2} + \frac{1}{r} \frac{dA_\theta}{dr} + \left(-j\omega \mu_o \sigma - \frac{1}{r^2} \right) A_\theta = 0$$

The solution is (6)

$$A_\theta = a J_1 \left[(-j\omega \mu_o \sigma)^{1/2} r \right] \quad (4)$$

since A_θ is finite at $r = 0$. We thus have

$$\frac{dA_\theta}{dr} = a \left[k J_0(kr) - \frac{J_1(kr)}{r} \right]$$

with

$$k = (-j\omega \mu_o \sigma)^{1/2}$$

Outside of the core, the vector potential is that of a uniform field. (See sec. II.)

$$A_\theta = \frac{B_o r}{2} + \frac{c}{r}$$

Matching value and slope at $r = a$, we obtain

$$C = \frac{B_o a^2}{2} \left[\frac{2J_1(ka)}{ka J_0(ka)} - 1 \right]$$

$$a = \frac{\frac{B_o a}{2} + \frac{c}{a}}{J_1(ka)}$$

To calculate the flux linking a turn of the coil secondary, we must calculate the line integral of the vector potential,

$$\int \vec{A} \cdot d\vec{s}$$

around the coil turn. If no core were present, we should still have the term $B_o r/2$. Therefore, the change in flux linkage due to the presence of the core is

$$\begin{aligned}\Delta\phi &= \int \frac{c}{r_{\text{coil}}} \cdot ds = \frac{c}{r_{\text{coil}}} \cdot 2\pi r_{\text{coil}} = 2\pi c \\ &= \pi B_o a^2 \left[\frac{2J_1(ka)}{J_o(ka)} - 1 \right]\end{aligned}$$

Now the mutual inductance M is given by $M = N_s \phi/I$; and also, $B_o = \mu_o n I$ for a long solenoid, where n is the number of turns per unit of length in primary, and N_s is the number of turns in secondary. Therefore,

$$(\Delta M)_\infty = \frac{\Delta\phi N_s}{I} = \pi\mu_o N_s na^2 \left[\frac{2J_1(ka)}{J_o(ka)} - 1 \right]$$

where the subscript ∞ signifies that we are dealing with an infinite core and coil. If we define

$$\frac{2J_1(ka)}{ka J_o(ka)} = M'(ka)$$

then

$$(\Delta M)_\infty = \pi\mu_o N_s na^2 [M'(ka) - 1] \quad (5)$$

Following McLachlan (7), we may write the complex function $J_\nu(ka)$ in the polar form:

$$J_\nu(mj^{3/2}) = M_\nu(m) \exp[j\theta_2(m)]$$

and thus

$$\begin{aligned}(\Delta M)_\infty &= \pi\mu_o N_s na^2 \left\{ \frac{2M_1(ma)}{ka M_o(ma)} \exp[j(\theta_1 - \theta_o)] - 1 \right\} \\ &= \pi\mu_o N_s na^2 \left\{ \frac{2M_1(ma)}{ma M_o(ma)} \exp\left[j\left(\theta_1 - \theta_o - \frac{3\pi}{4}\right)\right] - 1 \right\}\end{aligned}$$

with $m = (\mu_o \sigma \omega)^{1/2}$; or, in terms of real and imaginary parts,

$$\begin{aligned}(\Delta M)_\infty &= \pi\mu_o N_s na^2 \left\{ \left[\frac{2M_1(ma)}{ma M_o(ma)} \sin\left(\theta_1 - \theta_o - \frac{\pi}{4}\right) - 1 \right] \right. \\ &\quad \left. - j \left[\frac{2M_1(ma)}{ma M_o(ma)} \cos\left(\theta_1 - \theta_o - \frac{\pi}{4}\right) \right] \right\}\end{aligned}$$

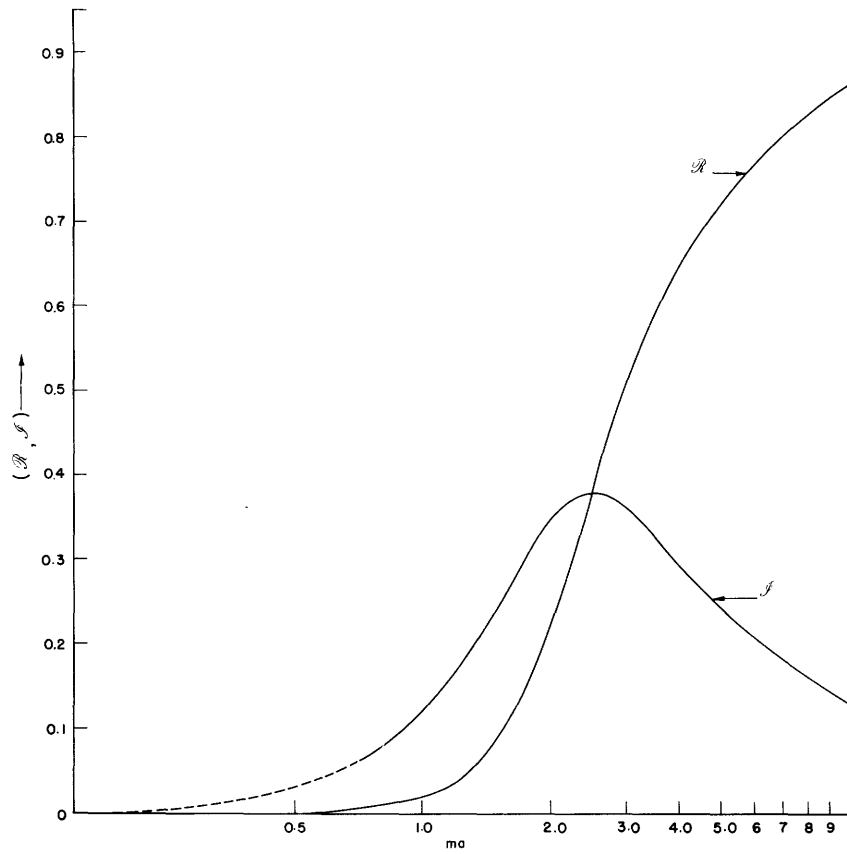


Fig. 3

The functions $M'_r(ma)$, $M'_1(ma)$, and the corrected functions \mathcal{R} and \mathcal{J} as functions of $ma = \sqrt{2}(a/\delta)$ with δ = the skin depth. The corrections are too small to be distinguished to the scale of the figure.

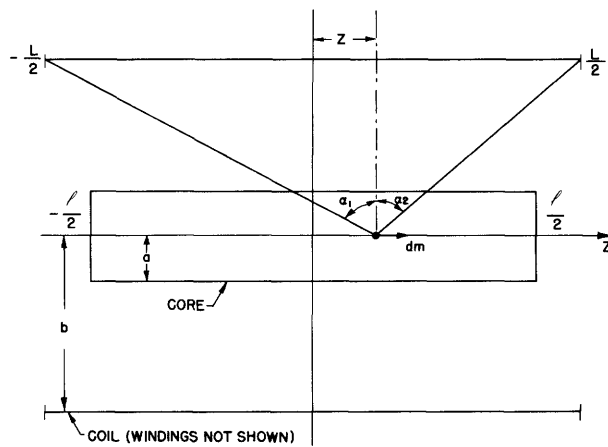


Fig. 4

A finite core within a finite coil.

Thus

$$\frac{(\Delta M)_\infty}{\pi \mu_0 N_s n a^2} = M'_r(\text{ma}) - j M'_i(\text{ma}) \quad (6)$$

The curves of $M'_r(\text{ma})$ and $M'_i(\text{ma})$ are given in Fig. 3 (8). To the scale of the figure, they are indistinguishable from the curves of \mathcal{R} and \mathcal{I} , which are discussed later.

B. The Case of a Finite Core in a Finite Coil

1. The dipole term

We now take up the case of a finite core in a finite coil. We shall use the multipole expansion of the vector potential derived in section II. The calculation may be sketched as follows: We first determine the current density, \vec{J} in the core. An approximate expression for \vec{J} is used in which the finite length of the coil is taken into account. We replace the current distribution by an equivalent set of magnetic multipoles along the axis of the core. The flux due to this finite multipole line which links the secondary coil is then calculated. From the flux linkage, we obtain an expression for ΔM .

Figure 4 shows the finite core and coil to be considered. The following notation will be used:

a = core radius	ℓ = core length	s subscript refers to secondary
b = coil radius	L = coil length	p subscript refers to primary

We first calculate the contribution due to the dipole term in the vector potential $\vec{A}^{(1)}$. The flux which links the secondary due to a dipole \vec{dm} located on the coil axis is calculated as follows (see Fig. 4): From Eq. 2 we obtain the expression for the vector potential of a dipole

$$\vec{A} = \frac{\mu_0 dm}{4\pi r^2} \sin \theta \vec{i}_\phi$$

Therefore, for a turn at x the flux linkage is

$$d\phi = \int \vec{A} \cdot d\vec{s} = \frac{\mu_0 dm}{4\pi r^2} \sin \theta \cdot 2\pi b$$

But $\sin \theta = \cos \alpha$, and $r = b/\cos \alpha$. Therefore,

$$d\phi = \frac{\mu_0 dm}{2b} \cdot \cos^3 \alpha$$

Now, $d(Nd\phi) = nd\phi dx$, where n is the number of turns per unit length on the coil, $d\phi$ is the differential of flux linkage due to a section dx of the coil. Thus,

$$Nd\phi = \int_{\text{coil}} \frac{\mu_0 dm}{2b} n \cos^3 \alpha dx$$

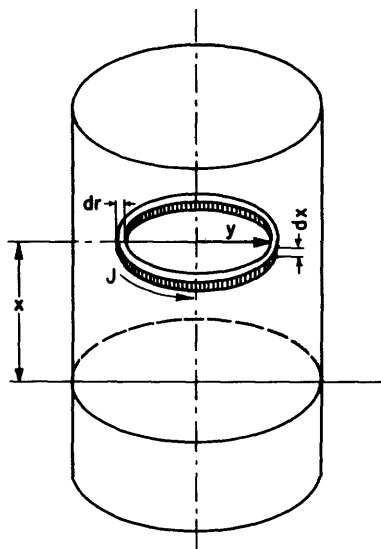


Fig. 5

Construction used for the calculation of the dipole moment equivalent to a circular current distribution.

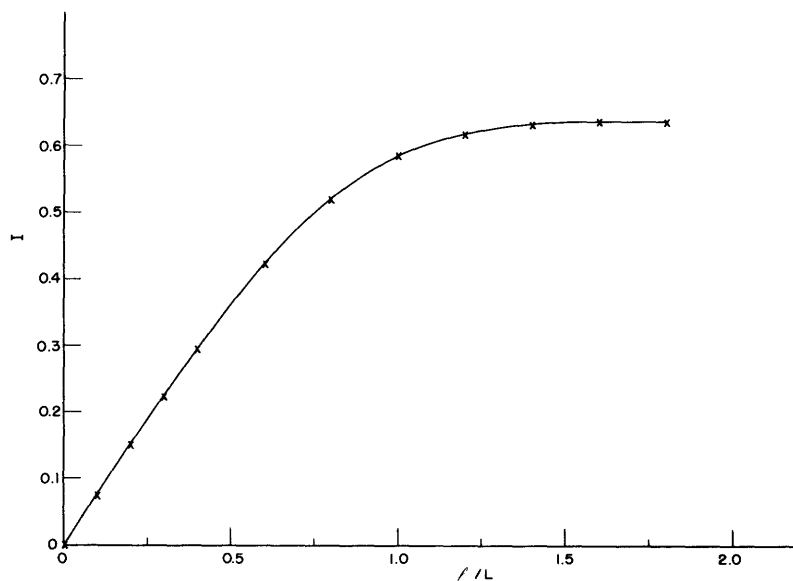


Fig. 6

The geometrical factor, $I(K, \frac{l}{L})$ as a function of $\frac{l}{L}$.

But $x = b \tan \alpha$; $dx = b \sec^2 \alpha d\alpha$. Therefore

$$\begin{aligned} d(N\phi) &= \int_{\text{coil}} \frac{\mu_o dm}{2} n \cos \alpha d\alpha \\ &= \frac{n\mu_o dm}{2} (\sin \alpha_2 - \sin \alpha_1) \end{aligned} \quad (7)$$

which is the required relation.

Referring to Stratton (9), we may write an expression for the field B_o along the axis of the coil (with the core absent)

$$B_o = \frac{\mu_o n_p I}{2} (\sin \alpha_2 - \sin \alpha_1) \quad (8)$$

With the core present, we assume that the vector potential is given by Eq. 4

$$A_\theta = \alpha J_1(kr)$$

with

$$\alpha = \frac{B_o a}{2} \frac{M'(ka)}{J_1(ka)}$$

We shall assume that B_o is given by the relation in Eq. 8 and use the expression, $\vec{J} = -j\omega\sigma \vec{A}$. We shall also need the expression for the equivalent dipole moment of a current distribution. We shall assume that the current flow is cylindrical, that is to say, that \vec{J} depends only upon r and x . (See Fig. 5.) We may write immediately

$$d^2m = d^2I \cdot A = (J dr dx) \cdot \pi r^2$$

or

$$d^2m = \pi J r^2 dr dx$$

which is the expression for the dipole moment equivalent to the current loop pictured in Fig. 5. If we integrate over r , we obtain

$$dm = \int_{r=0}^{r=a} \pi r^2 J dr dx$$

We then find

$$\begin{aligned} dm &= \frac{\pi B_o a^2}{\mu_o} [M'(ka) - 1] dx \\ &= \frac{\pi}{\mu_o} a^2 \frac{\mu_o n_p I}{2} (\sin \alpha_2 - \sin \alpha_1) [M'(ka) - 1] dx \end{aligned}$$

Putting this in the relation of Eq. 7, we obtain

$$\frac{d(N\phi)}{I} = \frac{n_s \mu_o}{2} \frac{\pi a^2 n_p}{2} [M'(ka) - 1] (\sin \alpha_2 - \sin \alpha_1)^2 dx$$

or, integrating over the core length ℓ , as shown in Fig. 4

$$\Delta M = \frac{N\phi}{I} = \frac{\pi}{4} a^2 \mu_o n_s n_p [M'(ka) - 1] \int_{-\ell/2}^{\ell/2} (\sin \alpha_2 - \sin \alpha_1)^2 dx$$

Referring to Fig. 4, we see that

$$\sin \alpha_2 = \frac{\frac{L}{2} - x}{\left[\left(\frac{L}{2} - x \right)^2 + b^2 \right]^{1/2}}$$

$$\sin \alpha_1 = \frac{-\left(\frac{L}{2} + x \right)}{\left[\left(\frac{L}{2} + x \right)^2 + b^2 \right]^{1/2}}$$

We may therefore write this integral as

$$\int_{-\ell/2}^{\ell/2} (\sin \alpha_2 - \sin \alpha_1)^2 dx = \int_{-\ell/2}^{\ell/2} \left\{ \frac{\left(x - \frac{L}{2} \right)}{\left[\left(x - \frac{L}{2} \right)^2 + b^2 \right]^{1/2}} - \frac{\left(x + \frac{L}{2} \right)}{\left[\left(x + \frac{L}{2} \right)^2 + b^2 \right]^{1/2}} \right\}^2 dx$$

or, defining

$$I = \frac{1}{4L} \int_{-\ell/2}^{\ell/2} \left\{ \frac{\left(x - \frac{L}{2} \right)}{\left[\left(x - \frac{L}{2} \right)^2 + b^2 \right]^{1/2}} - \frac{\left(x + \frac{L}{2} \right)}{\left[\left(x + \frac{L}{2} \right)^2 + b^2 \right]^{1/2}} \right\}^2 dx$$

we may transform this integral in the following way: Put

$$z = \frac{2x}{L}; \quad K = \frac{2b}{L} = \frac{D}{L}$$

Then

$$I\left(K, \frac{\ell}{L}\right) = \frac{1}{4L} \int_{-\ell/L}^{\ell/L} \left\{ \frac{(z - 1)}{[(z - 1)^2 + K^2]^{1/2}} - \frac{(z + 1)}{[(z + 1)^2 + K^2]^{1/2}} \right\}^2 \cdot \frac{L}{2} dz$$

$$= \frac{1}{4} \int_0^{\ell/L} \left\{ \frac{(z - 1)}{[(z - 1)^2 + K^2]^{1/2}} - \frac{(z + 1)}{[(z + 1)^2 + K^2]^{1/2}} \right\}^2 dz$$

This integral may be evaluated numerically for a given value of K . Figure 6 shows a plot of this integral evaluated for $K = 0.5625$. Thus

$$\Delta M = \pi a^2 \mu_o N_s n_p \left[M'(ka) - 1 \right] \cdot L \cdot I\left(K, \frac{\ell}{L}\right)$$

Referring to Eq. 6, we obtain, finally,

$$\frac{\Delta M}{(\Delta M)_\infty} = I\left(K, \frac{\ell}{L}\right) \quad (9)$$

2. The higher multipole terms

As a further refinement to the calculation, we shall include the octapole contribution to the vector potential, $\vec{A}^{(3)}$. It was shown in section II that the quadrupole contribution from a circular filament is zero. The octapole contribution, however, is not zero, and we may write for this term (see Fig. 2)

$$\vec{A}^{(3)} = \frac{3}{8\pi} \frac{\mu_o I r'^4}{r^4} \sin \theta \left[\frac{5}{4} \sin^2 \theta - 1 \right] \vec{i}_y$$

or, if $m = \pi r'^2 I$

$$\vec{A}^{(3)} = \frac{3}{8\pi} \frac{\mu_o m r'^2}{r^4} \sin \theta \left[\frac{5}{4} \sin^2 \theta - 1 \right] \vec{i}_y$$

The flux linkage due to this potential at the turn at P is

$$\begin{aligned} \Delta \phi &= \oint \vec{A} \cdot d\vec{s} = \frac{3\mu_o m}{8\pi} \int \frac{r'^2}{r^4} \left[\frac{5}{4} \sin^2 \theta - 1 \right] \sin \theta \, ds \\ &= \frac{3\mu_o m}{8\pi} \frac{r'^2}{r^4} \left[\frac{5}{4} \sin^2 \theta - 1 \right] \sin \theta \cdot 2\pi b \end{aligned}$$

Referring to Fig. 7, $r = b/\sin \theta$.

Therefore

$$\Delta \phi = \frac{3\mu_o m}{4} \frac{r'^2}{b^3} \left[\frac{5}{4} \sin^2 \theta - 1 \right] \sin^5 \theta$$

If we now use $d(N\Delta\phi) = n_s \Delta\phi \, dx$, with $x = b \cot \theta$, $dx = -b \csc^2 \theta \, d\theta$, we obtain

$$\begin{aligned} N\Delta\phi &= \int_{\text{coil}} n_s \frac{3\mu_o m}{4} \frac{r'^2}{b^3} \left[\frac{5}{4} \sin^2 \theta - 1 \right] \sin^5 \theta \left[-b \csc^2 \theta \right] d\theta \\ &= -n_s \frac{3\mu_o m}{4} \frac{r'^2}{b^2} \int_{\theta_1=(\pi/2)-\alpha_1}^{\theta_2=(\pi/2)-\alpha_2} \left[\frac{5}{4} \sin^2 \theta - 1 \right] \sin^3 \theta \, d\theta \end{aligned}$$

or, performing the integration,

$$N\Delta\phi = -\frac{3}{4} n_s \mu_o m \frac{r'^2}{b^2} \cdot \frac{1}{4} \left[\cos^4 \alpha_1 \sin \alpha_1 - \cos^4 \alpha_2 \sin \alpha_2 \right]$$

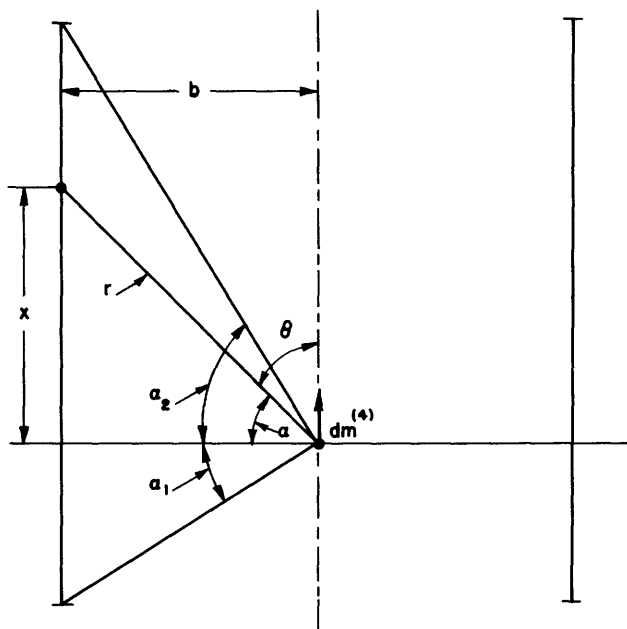


Fig. 7

Construction for the calculation of the flux linking a finite coil due to a magnetic multipole on the axis.

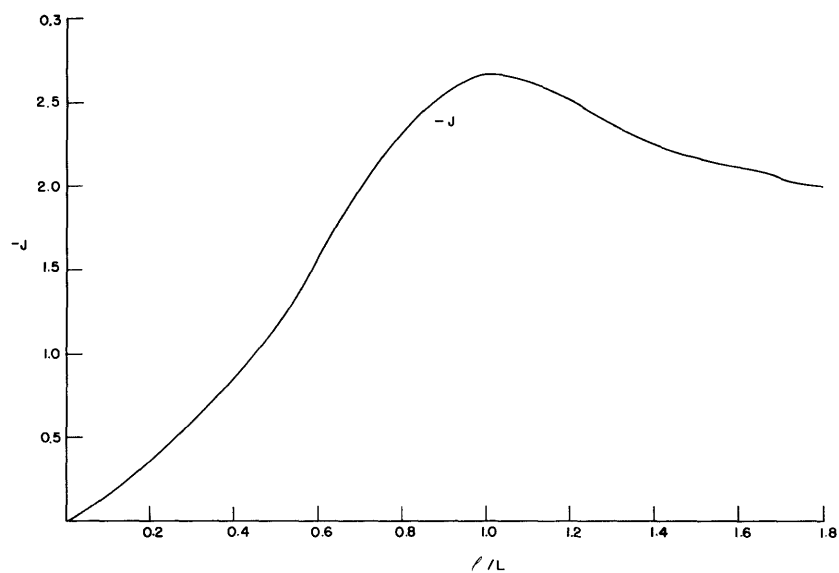


Fig. 8

The geometrical factor $J\left(K, \frac{l}{L}\right)$ as a function of $\frac{l}{L}$.

Therefore, we have as the addition to the change in mutual inductance, due to the octapole term,

$$(\Delta M)_{\text{octapole}} = \frac{(N\Delta\phi)_{\text{octapole}}}{I}$$

If we use the previous expression for $N\Delta\phi$

$$\frac{d(N\Delta\phi)}{I} = \frac{n\mu_o}{2I} \frac{dm}{b^2} \frac{r'^2}{8} \left[\cos^4 \alpha_1 \sin \alpha_1 - \cos^4 \alpha_2 \sin \alpha_2 \right]$$

Referring now to page 12, we obtain an expression for $dm \cdot r'^2$

$$dm \cdot r'^2 = \int_0^a \pi r^4 |\vec{J}| dr dz$$

with

$$|\vec{J}| = -j\omega\sigma |\vec{A}| = -j\omega\sigma a J_1(kr)$$

and

$$a = \frac{B_o}{2} \frac{M'(ka)}{J_1(ka)}$$

Therefore

$$dm \cdot r'^2 = \frac{-j\omega\pi\sigma B_o a}{2} \frac{M'(ka)}{J_1(ka)} dz \int_0^a J_1(kr) r^4 dr$$

Using

$$\int_0^a J_1(kr) r^4 dr = \frac{a^3}{k} \left[a J_2(ka) - \frac{2}{k} J_3(ka) \right]$$

and

$$B_o = \frac{\mu_o n_p I}{2} (\sin \alpha_2 - \sin \alpha_1)$$

we obtain

$$\begin{aligned} (\Delta M)_{\text{oct}} &= \frac{N\Delta\phi}{I} = \int_{\text{current loops}} \frac{d(N\Delta\phi)}{I} \\ &= \frac{n_s \mu_o}{2b^2} \frac{3}{8} \left(-\frac{j\omega\pi\sigma a}{2} \right) \frac{M'(ka)}{J_1(ka)} \frac{\mu_o n_p a^3}{2k} \left[a J_2(ka) - \frac{2}{k} J_3(ka) \right] \cdot A(\alpha, z) \end{aligned}$$

with (see Fig. 4)

$$A(\alpha, z) = \int_{-\ell/2}^{\ell/2} (\sin \alpha_2 - \sin \alpha_1) (\cos^4 \alpha_1 \sin \alpha_1 - \cos^4 \alpha_2 \sin \alpha_2) dz$$

Using

$$J_{p+1}(x) = \frac{2p}{x} J_p(x) - J_{p-1}(x), \quad 8M'(ka) = \frac{2J_1(ka)}{ka J_0(ka)}$$

we obtain

$$(\Delta M)_{\text{oct}} = \frac{3}{32} n_s n_p \mu_o \pi a^2 \left(\frac{a}{b}\right)^2 \left\{ 2M'(ka) - 2 + 1 - \frac{8}{(ka)^2} [M'(ka) - 1] \right\} A(a, z)$$

or

$$(\Delta M)_{\text{oct}} = \frac{3}{16} n_s n_p \mu_o \pi a^2 \left(\frac{a}{b}\right)^2 \left\{ M'(ka) - 1 + \frac{1}{2} \left[1 - \frac{8}{(ka)^2} (M'(ka) - 1) \right] \right\} A(a, z)$$

To evaluate $A(a, z)$, set

$$A(a, z) = \left(\frac{2b}{L}\right)^4 \cdot L J\left(K, \frac{\ell}{L}\right)$$

and let

$$\begin{aligned} \sin \alpha_2 &= \frac{\frac{L}{2} - x}{\left[\left(\frac{L}{2} - x\right)^2 + b^2 \right]^{1/2}} & \sin \alpha_1 &= \frac{-\left(\frac{L}{2} + x\right)}{\left[\left(\frac{L}{2} + x\right)^2 + b^2 \right]^{1/2}} \\ \cos \alpha_2 &= \frac{b}{\left[\left(\frac{L}{2} - x\right)^2 + b^2 \right]^{1/2}} & \cos \alpha_1 &= \frac{b}{\left[\left(\frac{L}{2} + x\right)^2 + b^2 \right]^{1/2}} \\ K &= \frac{2b}{L} = \frac{D}{L} & w &= \frac{2z}{L} \end{aligned}$$

Then

$$\begin{aligned} J\left(K, \frac{\ell}{L}\right) &= \int_0^{\ell/L} \left\{ \frac{(w+1)}{\left[(1+w)^2 + K^2 \right]^{1/2}} - \frac{(w-1)}{\left[(w-1)^2 + K^2 \right]^{1/2}} \right\} \\ &\quad \left\{ \frac{(w-1)}{\left[(w-1)^2 + K^2 \right]^{5/2}} - \frac{(w+1)}{\left[(w+1)^2 + K^2 \right]^{5/2}} \right\} dw \end{aligned}$$

The integral $J(k, \ell/L)$ can be evaluated numerically. The result of the integration is shown graphically in Fig. 8 for an integration over w in steps of 0.1 with $K^2 = 0.3164$. The expression for the change in mutual inductance can therefore be written

$$\begin{aligned} (\Delta M)_{\text{oct}} &= \frac{3}{16} n_s n_p \mu_o \pi a^2 \left(\frac{a}{b}\right)^2 \left\{ M'(ka) - 1 + \frac{1}{2} \left[1 - \frac{8}{(ka)^2} (M'(ka) - 1) \right] \right\} \\ &\quad \cdot L \cdot \left(\frac{2b}{L}\right)^4 J\left(K, \frac{\ell}{L}\right) \end{aligned}$$

or

$$\frac{(\Delta M)_{\text{oct}}}{\pi a^2 \mu_o n_s n_p L} = 3 \left(\frac{a}{b} \right)^2 \left(\frac{b}{L} \right)^4 \left\{ M'(ka) - 1 + \frac{1}{2} \left[1 - \frac{8}{(ka)^2} (M'(ka) - 1) \right] \right\} J \left(K, \frac{\ell}{L} \right)$$

Let us now combine the results of the dipole calculation with those of the octapole calculation. Summarizing

$$(\Delta M)_{\text{dip}} = \beta \left\{ [M'(ka) - 1] I \left(K, \frac{\ell}{L} \right) \right\}$$

$$(\Delta M)_{\text{oct}} = \beta \left\{ 3 \left(\frac{a}{b} \right)^2 \left(\frac{b}{L} \right)^4 \left\{ M'(ka) - 1 + \frac{1}{2} \left[1 - j \frac{8}{(ma)^2} (M'(ka) - 1) \right] \right\} J \left(K, \frac{\ell}{L} \right) \right\}$$

with

$$\beta = \pi \mu_o N_s n_p a^2$$

If we set

$$\left\{ M'(ka) - 1 + \frac{1}{2} \left[1 - j \frac{8}{(ma)^2} (M'(ka) - 1) \right] \right\} = \mathcal{R}' - j \mathcal{I}'$$

and use

$$M'(ka) - 1 = M'_r - j M'_i$$

Then

$$\mathcal{R}' = \frac{1}{2} + M'_r - \frac{4}{(ma)^2} M'_i$$

$$\mathcal{I}' = -M'_i - \frac{4}{(ma)^2} M'_r$$

Plots of \mathcal{R}' and \mathcal{I}' as functions of ma are given in Fig. 9. If we now label the observed decrease in the mutual inductance due to the presence of the core by $(\Delta M)_{\text{obs}}$ we may write

$$(\Delta M)_{\text{obs}} = (\Delta M)_{\text{dip}} + (\Delta M)_{\text{oct}}$$

Then

$$\begin{aligned} \frac{(\Delta M)_{\text{obs}}}{\beta I \left(K, \frac{\ell}{L} \right)} &= \frac{(\Delta M)_{\text{dip}}}{\beta I \left(K, \frac{\ell}{L} \right)} + \frac{(\Delta M)_{\text{oct}}}{\beta I \left(K, \frac{\ell}{L} \right)} \\ &= -M'_r + j M'_i + 3 \left(\frac{a}{b} \right)^2 \left(\frac{b}{L} \right)^4 (\mathcal{R}' + j \mathcal{I}') \frac{J \left(K, \frac{\ell}{L} \right)}{I \left(K, \frac{\ell}{L} \right)} \\ &= \mathcal{R} + j \mathcal{I} \end{aligned} \tag{10}$$

Table I

ma	$\mathcal{R} + M'_r$	$\mathcal{I} - M'_i$
0.7	0.00003	0.00009
1.0	0.00006	0.00027
1.5	0.00020	0.00056
2.0	0.00050	0.00083
3.0	0.0011	0.00096
4.0	0.0015	0.00092
5.0	0.0018	0.00089
6.0	0.0020	0.00085
7.0	0.0022	0.00081
8.0	0.0023	0.00076
9.0	0.0024	0.00072
10.0	0.0025	0.00068

For a typical case, we may use the following values of the variables:

$$\frac{\ell}{L} = 1$$

$$a = \frac{1}{4} \text{ inch}$$

$$L = 3.0 \text{ inch}$$

$$b = \frac{13}{16} \text{ inch}$$

Table I gives the quantities $\mathcal{R} + M'_r$ and $\mathcal{I} - M'_i$ for this case. Plots of the final function \mathcal{R} and \mathcal{I} are given in Fig. 3. Although they do not differ appreciably from M'_r and M'_i to the scale of the figure, the difference is significant in computations when tabulated values are used.

IV. Sensitivity Expressions

As a final calculation, we shall determine the sensitivity of the resistivity measuring system. This calculation will be performed with the use of the function M'_r and M'_i , since the functions \mathcal{R} and \mathcal{I} are too complicated for easy manipulation and are only small corrections. Since the slopes of M'_r and M'_i are nearly the same as the \mathcal{R} and \mathcal{I} functions, the results will be a good approximation for this case. We are interested in the change in the mutual inductance resulting from a fractional change in the resistivity. In particular, we define the sensitivity by

$$S = \frac{dM}{d\rho/\rho}$$

If we specify the slope of the $M'_r(\text{ma})$ line by $f(\text{ma})$, with

$$f(\text{ma}) = \frac{dM'_r}{f(\text{ma})}$$

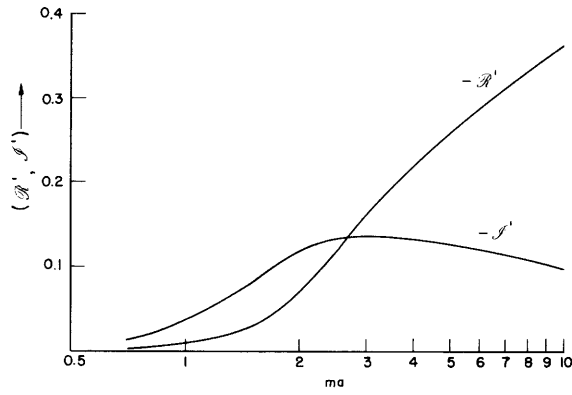


Fig. 9
The functions \mathcal{R}' and \mathcal{J}' as functions of ma .

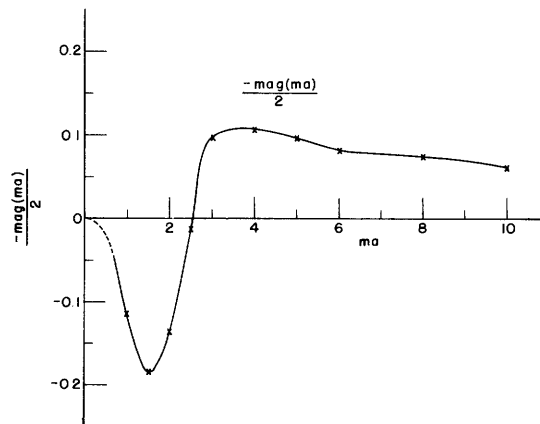
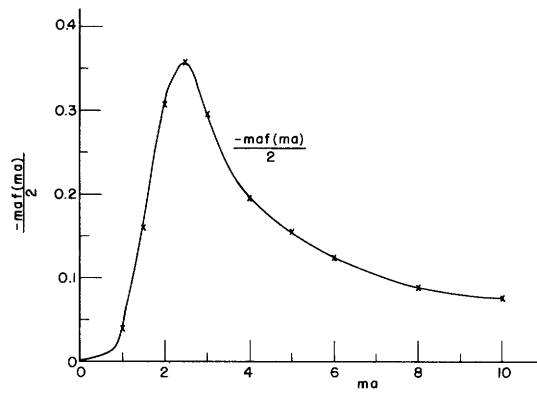


Fig. 10
The sensitivity functions $\frac{-maf(ma)}{2}$ and $\frac{-mag(ma)}{2}$ as functions of ma .

then at some value of ma , we have

$$d(ma) = \frac{dM'_r}{f(ma)}$$

as the change in ma corresponding to a change in M'_r . However

$$M'_r = \frac{\Delta M}{I \left(K, \frac{\ell}{L} \right)} \cdot \frac{1}{\pi \mu_o N_s n_p a^2} \quad (\text{see Eq. 9})$$

so that

$$dM'_r = \frac{d(\Delta M)}{I} \frac{1}{\pi \mu_o N_s n_p a^2}$$

Thus

$$d(ma) = \frac{d(\Delta M)}{I} \frac{1}{\pi \mu_o N_s n_p a^2 f(ma)}$$

Since

$$ma = \left(\mu_o \sigma \omega a^2 \right)^{1/2} = \left(\frac{\omega \mu_o a^2}{\rho} \right)^{1/2}$$

we have

$$d(ma) = -\frac{1}{2} \left(\frac{\omega \mu_o a^2}{\rho} \right)^{1/2} \frac{d\rho}{\rho} = \frac{d\rho}{\rho} \left(-\frac{1}{2} ma \right)$$

Therefore

$$\frac{d\rho}{\rho} = \frac{dM}{I \pi \mu_o N_s n_p a^2 \left[\frac{ma f(ma)}{2} \right]}$$

or

$$S_r = \frac{dM}{\frac{d\rho}{\rho}} = \pi \mu_o N_s n_p a^2 I \left(K, \frac{\ell}{L} \right) \left[\frac{ma f(ma)}{2} \right] \quad (11)$$

(We have written dM for $d(\Delta M)$. They are equivalent.)

Similarly, if S_i is wanted, we place

$$\frac{dM'_i}{d(ma)} = g(ma)$$

We obtain

$$S_i = \frac{dM}{\frac{d\rho}{\rho}} = \pi \mu_o N_s n_p a^2 I \left(K, \frac{\ell}{L} \right) \left[\frac{ma g(ma)}{2} \right] \quad (12)$$

It now remains to evaluate the quantities $f(ma)$ and $g(ma)$. To evaluate $f(ma)$, we use

$$\begin{aligned} M'_r &= \text{Re} \left[M'(ka) - 1 \right] \\ &= \text{Re} \left[\frac{2J_1(ka)}{ka J_0(ka)} - 1 \right] \end{aligned}$$

Thus

$$\frac{dM'_r}{d(ma)} = \frac{2}{ma} \operatorname{Re} \left[-\frac{2J_1(ka)}{ka J_0(ka)} + 1 + \frac{J_1^2(ka)}{J_0^2(ka)} \right]$$

or

$$\frac{-maf(ma)}{2} = \operatorname{Re} \left\{ M'(ka) - 1 + j \left(\frac{ma}{2} \right)^2 [M'(ka)]^2 \right\}$$

In a similar fashion

$$-M'_i = \operatorname{Im} [M'(ka) - 1]$$

and

$$\frac{mag(ma)}{2} = \operatorname{Im} \left\{ M'(ka) - 1 + j \left(\frac{ma}{2} \right)^2 [M'(ka)]^2 \right\}$$

Using

$$M'(ka) - 1 = M'_r - j M'_i$$

we obtain

$$\frac{-maf(ma)}{2} = M'_r + \frac{(ma)^2}{2} M'_i (1 + M'_r)$$

$$\frac{-mag(ma)}{2} = M'_i + \frac{(ma)^2}{4} [(1 + M'_r)^2 - (M'_i)^2]$$

Plots of $-maf(ma)/2$ and $-mag(ma)/2$ are given in Fig. 10.

Therefore, from a given value of $ma = (\mu_o \sigma \omega)^{1/2} a$ for a specimen, the values of the quantities $-maf(ma)/2$ and $-mag(ma)/2$ may be obtained from Fig. 10. Then Eqs. 11 and 12 give the sensitivities S_r and S_i .

V. Summary and Conclusions

We are now in a position to describe how one may calculate the conductivity σ of a metallic core from the observed value ΔM of the change in the mutual inductance obtained upon inserting the core into two concentric coils.

We designate by ΔM the change in the mutual inductance upon insertion of the core. Then, knowing the geometrical characteristics of the coils, we use Eq. 9 to determine $(\Delta M)_\infty$

$$(\Delta M)_\infty = \frac{\Delta M}{I \left(K, \frac{\ell}{L} \right)}$$

with the value of $I(K, \ell/L)$ taken from Fig. 6. Referring to Eq. 10, we can determine \mathcal{R} and \mathcal{I} , which are plotted in Fig. 3. From the plots of \mathcal{R} and \mathcal{I} , we can determine two values for the quantity ma , which should agree. From this value of ma , the conductivity can be determined, knowing the geometrical characteristics of the coil and core.

Any changes in the core conductivity will appear as changes in the mutual inductance of the coil. Therefore, the temperature dependence of the conductivity, for example, may be determined by following the mutual inductance as a function of temperature.

With a frequency of $33 \frac{1}{3}$ cps used in the mutual inductance bridge, resistivities of the order of 10^{-8} - 10^{-10} ohm-meters can be sensitively detected. This range is well suited for use in resistivity measurements in the temperature range below 50° K.

With the bridge described in Fig. 2, changes in mutual inductance of $1/2 \mu h$ can be observed. This gives a value of

$$\frac{\Delta \rho}{\rho} \approx 0.3 \text{ percent}$$

for the observable fractional change in resistivity.

Determinations of the value of the conductivity of the core are considered to be accurate to within approximately 15 percent. The values of conductivity found from the real and imaginary components of the mutual inductance have been found to agree within this limit. Measurements made at higher temperatures, where the conductivity is known, agree well with the known values. For very large values of the conductivity, when ma is close to 10, the agreement is not as good.

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